

An ODE-based approach to some Riemann–Hilbert problems motivated by wave diffraction

A. V. Shanin

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Abstract

A novel approach to Riemann–Hilbert problems of particular class is introduced. The approach is applicable to problems in which the multiplicative jump is set on a half-line. Such problems are linked to some Wiener–Hopf problems motivated by diffraction theory. The new approach is based on ordinary differential equations: the Riemann–Hilbert problem is reduced to finding a coefficient of an ordinary differential equation and solving this equation. The new method leads to an efficient numerical algorithm and opens a road to new asymptotical and analytical advances.

1 Introduction

Homogeneous Riemann–Hilbert problems are under consideration. A rigorous formulation of these problems and some important results related to them can be found in [1]. These problems are constituted in finding a vector function analytical (or meromorphic) on the complex plane and having a multiplicative jump on a smooth curve. Traditionally, if the curve coincides with the real axis the problem is referred to as the Wiener–Hopf factorization problem. Wiener–Hopf problems are directly linked with diffraction theory, hydrodynamics and some other branches. Numerous important canonical diffraction problems (such as diffraction by a half-plane and reflection by waveguide end) can be solved by the Wiener–Hopf method [2]. All scalar Wiener–Hopf problems and only some particular vector Wiener–Hopf problems can be solved explicitly. The solvable problems include a very important Khrapkov’s class [3, 4] admitting commutative factorization. For other problems some approximate techniques exist (see e.g. [5]). All problems studied in the papers mentioned above arise from either wave diffraction theory or solid mechanics.

In the current paper a novel technique is proposed to address the Riemann–Hilbert problems possibly not belonging to known factorizable classes. The approach is based on the ordinary differential equations. While it does not

provide explicit solutions, it opens some new possibilities for asymptotical and analytical studies of the Riemann-Hilbert problems. Also it leads to an efficient numerical algorithm.

The current paper continues a research started for the problems of wave scattering by a diffraction gratings. A general framework has been developed for these problems, comprising application of the so-called embedding formula, spectral equation and the problem of reconstruction of an ODE coefficient. The current work is dedicated to a possible generalization of the last step of the procedure.

2 The main result

2.1 Problem formulation

Consider the following matrix Riemann-Hilbert problem. Let straight contour C connect the point b with $b + i\infty$. The complex plane is cut along C . The left edge of the cut is denoted by C^- , and the right edge is denoted by C^+ .

Let there exist an open strip Ω of non-zero thickness embracing the cut C , such that the point b is located on the boundary of Ω (see Fig. 1). Let 2×2 matrix function $M(z)$ be analytical in Ω and continuous at $z = b$. Let also be

$$M(z) \rightarrow I \quad \text{as} \quad z \rightarrow b + i\infty, \quad (1)$$

where I is a 2×2 identity matrix.

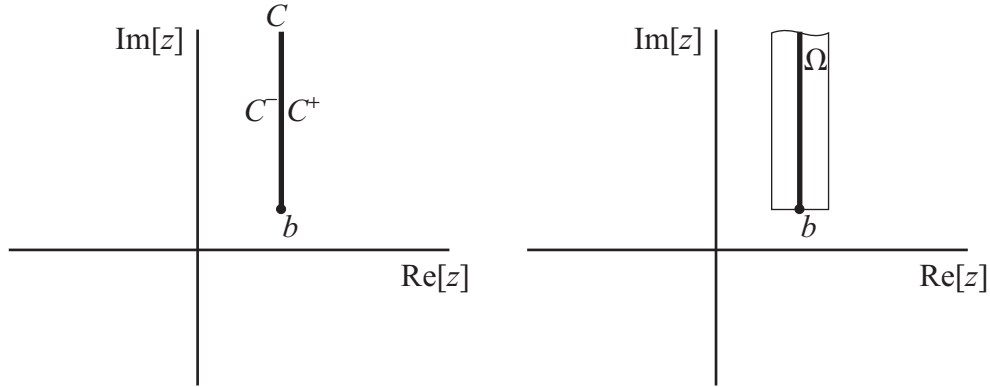


Figure 1: Contour C and domain Ω

The Riemann-Hilbert problem studied in this paper is formulated as follows. It is necessary to find 2×2 matrix function U analytical on the complex plane cut along contour C , having no zeros of the determinant on the cut plane, and obeying the following condition of the cut:

$$U^+(z) = U^-(z) M(z), \quad (2)$$

where $U^+(z)$ and $U^-(z)$ are the limiting values of the function U on the edges of the cut. Note that the right multiplication by M is used in problem formulation.

Additionally, demand that the function $U(z)$ tends to I as $|z| \rightarrow \infty$. Thus, a *canonical* solution is looked for (in Gakhov's sense, [1]). It is assumed also that U has no stronger than power growth near $z = b$.

2.2 Motivation of the problem

The problem formulation can be considered as a particular case of a standard Riemann–Hilbert problem set on a line going from $b + i\infty$ to $b - i\infty$. Namely, C is a part of this line, and M can be defined on the rest of the line as a unity matrix. However, the author has in mind another link to a standard matrix factorization problem. Namely, let b belong to the upper half-plane, and let the contour C' coincide with the real axis. Let the unknown matrix function U be analytical everywhere except C' and be discontinuous on the contour C' . Let the jump on the contour C' be described by the relation

$$U(z + 0i) = U(z - 0i) A(z), \quad z \in C' \quad (3)$$

where $A(z)$ is a known matrix coefficient. Finally, let A be an algebraic function of z defined on the whole complex plane z and having branch points only at $z = \pm b$. The last condition is quite strong, however this is not a rare occasion in the diffraction–motivated factorization problems.

Denote by \bar{U} the continuation of U from the lower half-plane. According to (3) this continuation is given by

$$\bar{U}(z) = U(z) A^{-1}(z), \quad \text{Im}[z] > 0, \quad (4)$$

i.e. the function $\bar{U}(z)$ can be continued onto the complex plane cut along C .

Denote by the upper indices \pm the values of \bar{U} and A related to the edges of the cut C . Obviously,

$$\bar{U}^+ = U(z) (A^+(z))^{-1}, \quad \bar{U}^- = U(z) (A^-(z))^{-1}. \quad (5)$$

Finally,

$$\bar{U}^+(z) = \bar{U}^-(z) A^-(z) (A^+(z))^{-1}, \quad z \in C \quad (6)$$

Thus, function \bar{U} obeys a “half-line” Riemann–Hilbert problem of the type introduced above with

$$M(z) = A^-(z) (A^+(z))^{-1}.$$

It is possible to reformulate a lot of practical diffraction problems as half–line Riemann–Hilbert problems.

2.3 “Ordered exponential” notations

Let $B(\tau)$ be a 2×2 matrix function analytical in some domain of complex plane. Define the *ordered exponential* of B (denoted by $\text{OE}[B]$) as a solution of differential equation

$$\frac{\partial}{\partial \tau_2} \text{OE}[B](\tau_2, \tau_1) = B(\tau_2) \cdot \text{OE}[B](\tau_2, \tau_1) \quad (7)$$

with initial condition

$$\text{OE}[B](\tau_1, \tau_1) = I.$$

Here τ_1 and τ_2 belong to the domain of analyticity of $B(\tau)$. Obviously, in this case $\text{OE}[B](\tau_2, \tau_1)$ does not depend on the contour along which the ordinary differential equation is solved.

If a large area of variation of τ_2 is considered, in which $B(\tau_2)$ can have poles, it is necessary to indicate a contour γ connecting τ_1 and τ_2 , along which equation (7) is solved. In this case the notation looks like $\text{OE}[B](\gamma)$.

Obvious properties of the ordered exponential are as follows:

$$\text{OE}[B](\gamma_1 + \gamma_2) = \text{OE}[B](\gamma_2) \cdot \text{OE}[B](\gamma_1), \quad (8)$$

$$\text{OE}[B](-\gamma) = (\text{OE}[B](\gamma))^{-1}. \quad (9)$$

Here and below $\gamma_1 + \gamma_2$ denotes concatenation of the contours, $-\gamma$ corresponds to the contour γ passed in the opposite direction.

2.4 Formulation of the main result of the paper

Theorem 1 *Let $r(\tau)$ be a function defined and analytical in Ω and continuous at $\tau = b$ such that for all points $z \in (C \setminus b)$ the following equation is valid:*

$$\text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (\gamma_z) = M(z), \quad (10)$$

where contour γ_z (shown in Fig. 2) lies within Ω . z in the left of (10) plays role of a fixed parameter, while τ is the independent variable. Then solution of the Riemann–Hilbert problem formulated above is given by the formula

$$U(z) = \text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (b, b + i\infty), \quad (11)$$

where the ordinary differential equation is solved along contour C .

PROOF

Obviously, relation (11) defines a function analytical on the complex plane cut along contour C . Thus, it is necessary to prove (2). Let be $z \in C$. Deform contour γ_z as it is shown in Fig. 3. The contour becomes a concatenation of γ_-

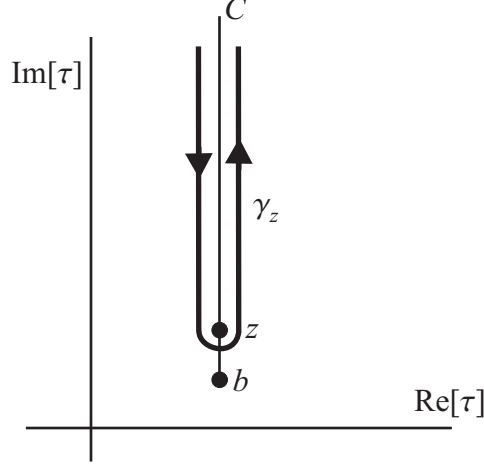


Figure 2: Contours for the OE equation

going from $+i\infty$ to b along the left edge of the cut, and the contour γ_+ going from b to $+i\infty$ along the right edge of the cut. According to (8), (9), and (10),

$$\left(\text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (-\gamma_+) \right)^{-1} \text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (\gamma_-) = M(z). \quad (12)$$

However, according to (11)

$$\text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (\gamma_-) = U^+(z), \quad \text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (-\gamma_+) = U^-(z), \quad (13)$$

thus (2) is proven \square

3 Interpretation of the main result

Although the proof of Theorem 1 is simple and straightforward, it would be useful to demonstrate another interpretation of this theorem. Let β be a point on contour C , and let C_β be a part of C going from β to $+i\infty$. Formulate a family of Riemann–Hilbert problems, indexed by β , having cut C_β instead of C (all other conditions of the problem remain the same). The coefficient of each problem belonging to the family is a restriction of function $M(z)$ on contour C_β . Note that M does not depend on β .

Denote by $U(\beta; z)$ the family of solutions for these problems. Consider the derivative

$$W(\beta; z) = \frac{\partial}{\partial \beta} U(\beta; z). \quad (14)$$

$W(\beta; z)$ taken as a function of z is isomonodromical to $U(\beta; z)$ in the following sense: it is analytical on the plane with the same cut C_β , and the multiplicative

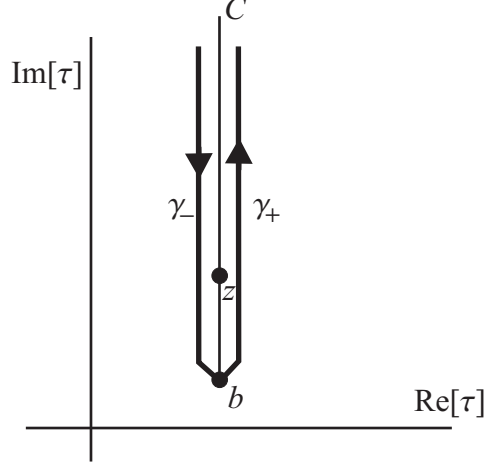


Figure 3: Contours γ_+ and γ_-

jump on this cut is described by the same coefficient $M(z)$. Moreover, for each β function $W(\beta; z)$ decays as $|z| \rightarrow \infty$, and this function grows no stronger than a power function near $z = \beta$.

The isomonodromy between W and U can be used as follows. One can prove that

$$W(\beta; z) = R(\beta; z) U(\beta) \quad (15)$$

where $R(\beta; z)$ is a rational function of z for each fixed β . To prove this, define R as

$$R = W U^{-1}.$$

Obviously, R has no singularities on the plane cut along C_β , except maybe the point $z = \beta$. The discontinuity of W on C_β is described by the relation

$$\begin{aligned} R^+(\beta; z) &= W^+(\beta; z) (U^+(\beta; z))^{-1} = W^-(\beta; z) M(z) M^{-1}(z) (U^-(\beta; z))^{-1} = \\ &= W^-(\beta; z) (U^-(\beta; z))^{-1} = R^-(\beta; z), \end{aligned} \quad (16)$$

i.e. R is continuous on C_β . Thus, R is single valued, has a power singularity at $z = \beta$, and decays as $|z| \rightarrow \infty$. Therefore it is rational. A detailed study of the singularity at $z = \beta$ shows that R should have form

$$R(\beta; z) = \frac{r(\beta)}{z - \beta}. \quad (17)$$

Taking into account that the Riemann–Hilbert problem degenerates as $\beta \rightarrow b + i\infty$, and $U(b + i\infty; z) = I$, the representation (11) follows from (15) rewritten as a differential equation

$$\frac{\partial}{\partial \beta} U(\beta; z) = R(\beta; z) U(\beta; z). \quad (18)$$

Note 1. The method proposed here is based on embedding of the considered Riemann–Hilbert problem into a family having parameter β such that the differentiation with respect to the parameter provides an isomonodromy. In this context an elementary problem can be considered, for which M is a constant matrix with respect to z (or there are several cuts C_j and different constant matrices M_j defined on them). It is not necessary to embed this problem into a family of other problems, since differentiation d/dz provides an isomonodromy itself. The argument similar to the one provided above shows that U in this case is a solution of a Fuchsian equation, and M_j are the monodromy matrices of this equation.

Note 2. Variation of the end point of the cut is not the only possible way to organize a family, in which the differentiation with respect to a parameter provides an isomonodromy. Another way is to study matrix $M(\alpha; z)$ depending on some parameter α and construct a differential operator

$$H \equiv a_1(\alpha; z) \frac{\partial}{\partial \alpha} + a_2(\alpha; z) \frac{\partial}{\partial z}$$

such that $H[M] = 0$. Then it is possible prove that

$$H[U](\alpha; z) = R(\alpha; z) U(\alpha; z),$$

where R is a single-valued function with respect to z .

4 Examples

4.1 Scalar example

Consider a scalar example. Let it be necessary to find function $u(z)$ obeying the following restrictions:

- The function should be regular on the complex plane cut along the line $C = (b, b + i\infty)$.
- The values on the edges of the cut should be connected via the relation

$$u^+(z) = u^-(z) m(z), \tag{19}$$

where u^+ are the values on the right edge of the cut, u^- are the values on the left edge of the cut, $m(z)$, $z \in \Omega$ is known function tending to 1 at infinity.

- The function should grow no faster than algebraically at b and tend to 1 at infinity.

Obtain a “traditional” solution for this problem. Note that a multiplicative jump of u corresponds to an additive jump of $\log u$. Thus, the solution can be

obtained by using Cauchy's integral:

$$\log u(z) = -\frac{1}{2\pi i} \int_b^{b+i\infty} \frac{\log(m(\xi))}{\xi - z} d\xi. \quad (20)$$

Now construct a “new” solution for this problem. According to a scalar analog of Theorem 1, it is necessary to find a function $r(\tau)$ defined in Ω , such that an analog of (10) is fulfilled:

$$\text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (\gamma_z) = m(z), \quad (21)$$

This equation can be simplified using the fact that

$$\text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (\gamma_z) = \exp \left\{ \int_{\gamma_z} \frac{r(\tau)}{z - \tau} d\tau \right\}. \quad (22)$$

Thus one can find $r(\tau)$ explicitly:

$$r(\tau) = -\frac{\log(m(\tau))}{2\pi i}. \quad (23)$$

Using the scalar analog of (11)

$$u(z) = \exp \left\{ - \int_b^{b+i\infty} \frac{r(\tau)}{z - \tau} d\tau \right\}, \quad (24)$$

get the solution coinciding with (20).

4.2 Chebotarev–Khrapkov matrices

The most important class of matrices admitting explicit factorization is the class introduced by Chebotarev [6] and Khrapkov [3]. Later on, these matrices have been studied by numerous researchers. Among the most important works there are [7, 8, 9, 10].

Here the Khrapkov's 2×2 matrices are taken as an example. These matrices have the following form:

$$M(z) = c(z)I + p(z)L, \quad L(z) \equiv \begin{pmatrix} l(z) & m(z) \\ n(z) & -l(z) \end{pmatrix} \quad (25)$$

where $l(z)$, $m(z)$, $n(z)$ are some polynomials such that

$$f(z) \equiv l^2(z) + m(z)n(z) \quad (26)$$

is a polynomial of degree not higher than 2. It is assumed that $M \rightarrow I$ as $z \rightarrow b + i\infty$.

It is known that the Khrapkov's matrices can be factorized commutatively, but for the price of more flexible behaviour at infinity. Thus, the coefficient

$$R(\tau; z) = \frac{r(\tau)}{z - \tau}$$

in (11) and (10) should be replaced by a form not necessarily decaying as $|z| \rightarrow \infty$:

$$R(\tau; z) = \frac{\xi(\tau)}{z - \tau} I + \frac{\eta(\tau)}{z - \tau} L(z), \quad (27)$$

where ξ and η are some scalar functions.

The advantage of using the Khrapkov's matrices is their commutativity which leads to the closed-form formula for the ordered exponential:

$$\text{OE}[R(\tau; z)](\gamma) = \exp\{\hat{\xi}\} \left(\cosh\left(\sqrt{f(z)} \hat{\eta}\right) I + \sinh\left(\sqrt{f(z)} \hat{\eta}\right) \frac{L(z)}{\sqrt{f(z)}} \right), \quad (28)$$

where

$$\hat{\xi} = \int_{\gamma} \frac{\xi(\tau)}{z - \tau} d\tau, \quad \hat{\eta} = \int_{\gamma} \frac{\eta(\tau)}{z - \tau} d\tau, \quad (29)$$

Thus, in the analog of (10)

$$\text{OE}[R(\tau; z)](\gamma_z) = M(z) \quad (30)$$

contour γ_z can be changed to a small closed loop encircling the point z . According to this, (30) can be solved explicitly:

$$\xi(\tau) = \frac{i}{4\pi} \log(c^2(\tau) - f(\tau)p^2(\tau)), \quad (31)$$

$$\eta(\tau) = \frac{i}{4\pi\sqrt{f(\tau)}} \log\left(\frac{c(\tau) + p(\tau)\sqrt{f(\tau)}}{c(\tau) - p(\tau)\sqrt{f(\tau)}}\right). \quad (32)$$

Finally, the solution of the problem is as follows:

$$U(z) = \exp\{\bar{\xi}\} \left(\cosh\left(\sqrt{f(z)} \bar{\eta}\right) I + \sinh\left(\sqrt{f(z)} \bar{\eta}\right) \frac{L(z)}{\sqrt{f(z)}} \right), \quad (33)$$

where

$$\bar{\xi} = - \int_b^{b+i\infty} \frac{\xi(\tau)}{z - \tau} d\tau, \quad \bar{\eta} = - \int_b^{b+i\infty} \frac{\eta(\tau)}{z - \tau} d\tau. \quad (34)$$

Solution in the form (33) coincides with the one described in [3].

4.3 A link to Weinstein–class problems

The current paper can be considered as a generalization of a previous work by author dedicated to Weinstein–class problems. Problems of scattering by periodic diffraction gratings composed of absorbing screens have been studied. The first problem having the geometry shown in Fig. 4 a) is a classical Weinstein’s problem [11]. The period of the system consists of a single screen. The incident wave has wavenumber k_0 (such that $k_0 d \gg 1$) and a grazing incidence angle. The problem is considered in the parabolic approximation. In [12] this problem has been reformulated as the following scalar OE equation:

$$\text{OE}[r(\tau)/\sqrt{z-\tau}](\gamma_z) = 1 - e^{iz}, \quad (35)$$

where γ_z is the contour introduced above. This equation can be reduced to an integral equation with difference kernel and then solved explicitly. It is worth to note that this problem has been solved in [11] by using Wiener–Hopf method.

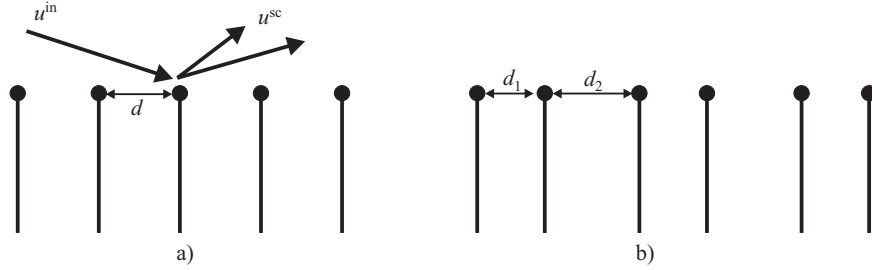


Figure 4: Geometry of Weinstein–class problems

In the second problem of Weinstein’s class (see Fig. 4 b)) the period of the grating contains two screens. In [13] this problem has been reduced to matrix OE equation

$$\text{OE} \left[\frac{r(\tau)}{\sqrt{z-\tau}} \right] (\gamma_z) = M(z), \quad (36)$$

$$M(z) = \begin{pmatrix} 1 & -\exp\{izd_2/(d_1 + d_2)\} \\ -\exp\{izd_1/(d_1 + d_2)\} & 1 \end{pmatrix}. \quad (37)$$

The same diffraction problem can be reduced to a Wiener–Hopf problem in which the matrix

$$K(\beta; y) = M(\beta - y^2(d_1 + d_2)/(2k_0)) \quad (38)$$

should be factorized with respect to variable y for any real value of parameter β .

Application of the equation (36) enabled one to construct an efficient numerical solution of the diffraction problem having geometry shown in Fig. 4 b. Moreover, using the asymptotic analysis of equation (36) it became possible to prove that the reflection coefficient for this structure tends to -1 as the angle on incidence tends to zero. This result is practically important since it guarantees high values of Q-factor in corresponding resonator problems [14].

5 Solving (10) numerically

5.1 Parametrization of equation (10)

Here equation (10) is addressed. It is shown below that the unknown matrix $r(\tau)$ depends on two unknown scalar parameters. These parameters are revealed here, and the problem for finding these parameters is formulated in the simplest.

Let the logarithm of the eigenvalues of M (and the eigenvalues themselves) be distinct everywhere on C . Represent M in the form

$$M(\tau) = T(\tau) \cdot \begin{pmatrix} \lambda_1(\tau) & 0 \\ 0 & \lambda_2(\tau) \end{pmatrix} \cdot T^{-1}(\tau) \quad (39)$$

Here T is the matrix whose columns are eigenvectors of M . Matrix T depends on two scalar parameters. Almost everywhere it can be parametrized as follows:

$$T(\tau) = \begin{pmatrix} 1 & 1 \\ t_1(\tau) & t_2(\tau) \end{pmatrix} \quad (40)$$

Values $\lambda_{1,2}$ and $t_{1,2}(\tau)$ are known.

The left-hand side of (10) can be rewritten as follows:

$$\text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (\gamma_z) = F \cdot \text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (\sigma_z) \cdot F^{-1} \quad (41)$$

where σ_z is a loop of a small radius ϵ starting and ending at $z + i\epsilon$ and encircling z in the positive direction, and

$$F = \text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (z + i\infty, z + i\epsilon).$$

Let $r(\tau)$ be represented in the form

$$r(\tau) = H(\tau) \begin{pmatrix} \zeta_1(\tau) & 0 \\ 0 & \zeta_2(\tau) \end{pmatrix} H^{-1}(\tau) \quad (42)$$

As $\epsilon \rightarrow 0$

$$\text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (\sigma_z) \rightarrow H(z) \cdot \exp \left\{ -2\pi i \begin{pmatrix} \zeta_1(z) & 0 \\ 0 & \zeta_2(z) \end{pmatrix} \right\} \cdot H^{-1}(z). \quad (43)$$

It follows from (40) that the eigenvalues of $r(z)$ are defined by the eigenvalues of $M(z)$:

$$\zeta_{1,2}(z) = \frac{i}{2\pi} \log(\lambda_{1,2}(z)). \quad (44)$$

Thus, to find r one needs only to find $H(\tau)$.

Again, the columns of $H(\tau)$ are the eigenvectors of r . Almost everywhere matrix H can be parametrized as follows:

$$H(\tau) = \begin{pmatrix} 1 & 1 \\ h_1(\tau) & h_2(\tau) \end{pmatrix} \quad (45)$$

So our aim is to find $h_{1,2}(\tau)$.

For $z, \beta \in C$, $\text{Im}[\beta] > \text{Im}[z]$ define the function

$$V(\beta; z) = \text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (\Gamma_{z,\beta}) \quad (46)$$

where contour $\Gamma_{z,\beta}$ is shown in Fig. 5. Obviously, V obeys the equation

$$\frac{\partial}{\partial \beta} V(\beta; z) = \frac{1}{z - \beta} (r(\beta) V(\beta; z) - V(\beta; z) r(\beta)). \quad (47)$$

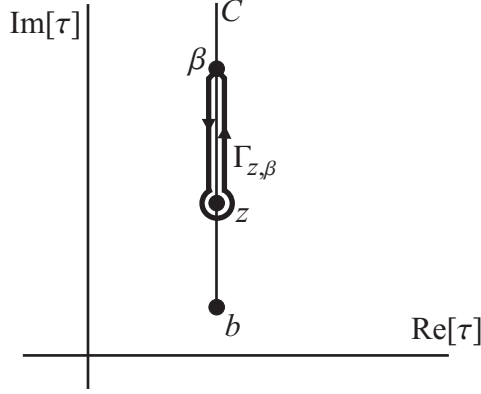


Figure 5: Contour $\Gamma_{z,\beta}$

Since V is adjoint to to

$$\text{OE} \left[\frac{r(\tau)}{z - \tau} \right] (\sigma_z),$$

its eigenvalues are the same, and it can be written as follows:

$$V(\beta; z) = Q(\beta; z) \cdot \begin{pmatrix} \lambda_1(z) & 0 \\ 0 & \lambda_2(z) \end{pmatrix} \cdot (Q(\beta; z))(z)^{-1}. \quad (48)$$

where

$$Q(\beta; z) = \begin{pmatrix} 1 & 1 \\ q_1(\beta; z) & q_2(\beta; z) \end{pmatrix}. \quad (49)$$

Taking $\beta \rightarrow z$ obtain the relation $Q(z; z) = H(z)$, and thus

$$q_{1,2}(z; z) = h_{1,2}(z). \quad (50)$$

Taking $\beta \rightarrow b + i\infty$ obtain $Q(\infty; z) = M(z)$, and thus

$$q_{1,2}(b + i\infty; z) = t_{1,2}(z). \quad (51)$$

Elementary calculations demonstrate that equation (47) is equivalent to the following system of differential equations:

$$\frac{\partial}{\partial \beta} q_{1,2}(\beta, z) = - \frac{(\zeta_1(\beta) - \zeta_2(\beta)) (q_{1,2}(\beta; z) - h_1(\beta)) (q_{1,2}(\beta; z) - h_2(\beta))}{(z - \beta) (h_1(\beta) - h_2(\beta))} \quad (52)$$

Note that these equations are independent from each other. Note also that they are Riccati equations. Finally, the reduced problem can be formulated as follows:

For given functions $\zeta_{1,2}(\tau)$, $t_{1,2}(\tau)$, $\tau \in C$ find functions $h_{1,2}(\tau)$, $\tau \in C$ such that for any $z \in C$ equation (52) possesses a solution $q_{1,2}(\beta, z)$ on the segment $\beta \in (z, b + i\infty)$ of C obeying boundary conditions (50), (51).

5.2 Numerical algorithm

The process of solving of the Riemann–Hilbert problem using the new method consists of two major stages. On the first stage the coefficient $r(\tau)$ is found on the contour C (it is shown above that it is sufficient to find two scalar functions $h_{1,2}(\tau)$). The coefficient is found by solving equation (10) or (52). On the second stage equation (11) is solved to find function $U(z)$ at some set of points z_k . The points at which the U should be computed do not necessarily belong to C . Moreover, in diffraction applications C is a half-line parallel to the imaginary axis, while it is necessary to find U on the real axis.

For the first stage contour C is meshed, i.e. a dense array of nodes τ_j is placed on it. Since the contour is infinite, it is necessary to choose a reasonable upper limit B instead of $b + i\infty$ (see Fig. 6). The nodes are numbered by indices $j = 1 \dots N$, where 1 corresponds to the node $\tau_1 = B$ (infinity), and index N corresponds to $\tau_N = b$.

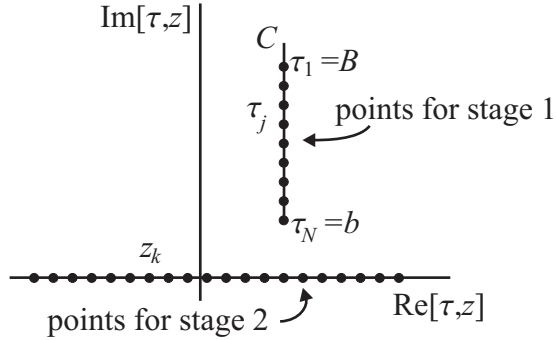


Figure 6: Nodes for numerical computations

At each τ_j matrix $M(\tau)$ is represented in the form (39), (40), i.e. the values $\lambda_{1,2}(\tau_j)$ and $t_{1,2}(\tau_j)$ are computed. The values $\zeta_{1,2}(\tau_j)$ are then computed by applying formula (44). The aim is to compute the values $h_{1,2}(\tau_j)$.

For the “infinity” point τ_1 the following values are assigned:

$$h_{1,2}(\tau_1) = t_{1,2}(\tau_1). \quad (53)$$

This is a natural choice for the asymptotics of the unknown coefficient, since $M(\tau) \rightarrow \mathbf{I}$ as $\tau \rightarrow b + i\infty$.

Then a loop over $j = 2 \dots N$ is performed. At the j th step of the loop the values $h_{1,2}(\tau_j)$ should be computed. Thus, when the j th step is performed all values $h_{1,2}(\tau_1) \dots h_{1,2}(\tau_{j-1})$ are already found. On the j th step of the loop equations (52) are solved on the segment (τ_1, τ_{j-1}) for $q_{1,2}$ using Runge–Kutta 4 method. The initial values are set as

$$q_{1,2}(\tau_1) = t_{1,2}(\tau_j) \quad (54)$$

following from (51). The values $q_{1,2}(\tau_{j-1})$ are found. Then equations (52) are solved on the segment (τ_{j-1}, τ_j) (only one step is performed) by Euler method. This method does not require the values of the right-hand side at the end of the segment to be known. The values $q_{1,2}(\tau_j)$ are found. The assignment

$$h_{1,2}(\tau_j) = q_{1,2}(\tau_j) \quad (55)$$

is performed, following from (50).

On the second stage for each z_k the array $r(\tau_j)/(z_k - \tau_j)$ is computed by (42), (45). Then the value

$$U(z_k) = \text{OE} \left[\frac{r(\tau)}{z_k - \tau} \right] (\tau_N, \tau_1)$$

is found by Runge–Kutta 4 method.

To demonstrate the feasibility of the new method the Khrapkov’s matrix of the simplest form has been taken:

$$M(z) = \mathbf{I} + \frac{1}{z^2} \begin{pmatrix} 1 & z \\ -z & -1 \end{pmatrix} \quad (56)$$

The matrix is set on contour $C = (2i, i\infty)$. The aim was to find the unknown function U at the points z_k belonging to the line $\text{Im}[z] = 1.8$.

The results obtained by the new method can be compared with the exact solution provided by Khrapkov’s formulae (33), (34). One has to take into account, however, that Khrapkov’s formulae lead to the solution $U_{\text{kh}}(z)$ that tends to some constant $U_\infty \neq \mathbf{I}$ as $|z| \rightarrow \infty$. Thus, the Khrapkov’s solution should be corrected as follows:

$$U = (U_\infty)^{-1} (U_{\text{kh}})(z) \quad (57)$$

For the computations the value of B has been chosen to $80i$, the step on the contour C was equal to 0.02. The results for $\text{Re}[U]_{1,1}$ and $\text{Re}[U]_{1,2}$ are shown in Fig. 7. Solid lines represent the Khrapkov’s solution corrected by formula (57), dots represent the solution obtained by applying the novel ODE-based method. One can see that the agreement between two exact formula and the new method is very nice.

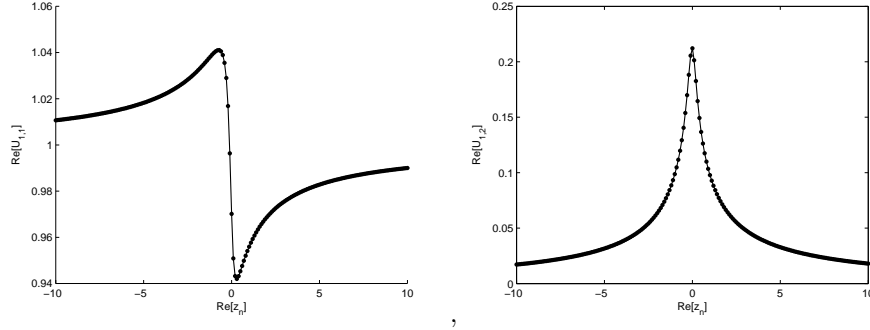


Figure 7: Results of numerical computations: $\text{Re}[U]_{1,1}$ (left), $\text{Re}[U]_{1,2}$ (right).

6 Conclusion

The paper presents a new analytical result concerning Riemann–Hilbert problem with the multiplicative jump set on a half-line. The solution can be represented in the form (11) with the unknown coefficient $r(\tau)$ obeying equation (10). Unfortunately, the new result does not lead to explicit solving of a general matrix factorization problem, but it provides an efficient numerical technique and gives some hopes to achieve further progress in asymptotical and analytical studies. The efficiency and robustness of the numerical method is linked with the fact that the equation (10) has the “Volterra” structure, i.e. the unknown coefficient can be found point-by-point. The asymptotical advantages of the new method has already been demonstrated for similar problems of Weinstein type. Finally, some analytical progress, hopefully, can be achieved by studying the analytical structure of the auxiliary function $r(\tau)$ for the case of rational or algebraic matrix M .

The new method brings the Riemann–Hilbert problem back into the context of the ordinary differential equations (from which it initially emerged), but in a slightly unusual form. Instead of studying the derivative with respect to the main variable z the derivative with respect to the parameter β is considered.

A new method is proposed for Riemann–Hilbert problems belonging to a particular class. As it is shown in the paper, the class naturally arises as a reformulation of a problem of factorization of a matrix A on the real axis if the matrix is given as an algebraic function having one branch point in the upper half-plane (or in the lower half-plane). Such case is quite common in the diffraction theory, but it restricts the class of problems to those having only one propagation speed. If there exist several wave modes with different speeds the new methods needs to be generalized to admit several half-infinite contours C_j . Such a generalization can be obtained by allowing the coefficient R to have several pole singularities.

The paper considers only matrices 2×2 , however most of the results can be easily converted for the case of matrix of arbitrary dimension.

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